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Hydrodynamic pressure on an accelerating dam and criteria for cavitation

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SUMMARY

The effect of finite reservoir on the hydrodynamic pressure due to horizontal as well as vertical ground excitations has been studied. It is found that for horizontal accelerations the hydrodynamic pressure force decreases as the size of the reservoir decreases. The effect of vertical acceleration on the pressure force on a dam is simply to adjust the hydrostatic pressure by replacing the gravitational constant by an effective gravitational acceleration and this is true for any arbitrary shapes of the reservoir. A simple criterion has been presented in this paper which would enable dam engineers to determine whether a given earthquake could cause cavitation at the dam-water interface or not.

1. Introduction

An accurate determination of the hydrodynamic pressure exerted on the up-stream face of a dam during earthquakes is important in the design of dams in seismic regions. During an earthquake a dam accelerates into and away from the water in the reservoir, and as a result, the water exerts a hydrodynamic pressure, in addition to the hydrostatic pressure, on the dam surface. Depending on the magnitude of the earthquake, this hydrodynamic pressure may, at certain points, exceed the hydrostatic pressure.

For an infinitely long reservoir, Westergaard [6] first derived an expression for the hydrodynamic pressure exerted on a dam by an incompressible, inviscid fluid in the reservoir as a result of horizontal harmonic ground motion in a direction perpendicular to the dam. He found that this hydrodynamic pressure is the same as if a certain body of fluid, often called the 'added mass', was forced to move back and forth with the dam. This added-mass concept has been used widely in fluid mechanics to calculate the increase of kinetic energy of an incompressible, inviscid fluid surrounding an accelerating solid body if the motion is irrotational. For a rigid dam with vertical up-stream face, Westergaard found that the added mass is confined in a volume bounded by a two-dimensional parabolic surface on the up-stream side of the dam. In a discussion to Westergaard's [6] paper, von Kármán [4] presented a simple momentum-balance method and obtained a distribution of the added mass, consequently the hydrodynamic pressure, along the vertical up-stream face of a rigid dam, very close to the Westergaard results. For

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rectangular, rigid reservoirs, Chopra [1] showed that the hydrodynamic forces due to the vertical component of ground motion are comparable to those due to horizontal ground motion.

By adopting a generalized version of von Kármán's momentum-balance method, Chwang and Housner [3] solved the two-dimensional problem of the added mass effect due to a horizontal acceleration of a rigid dam with an arbitrarily inclined up-stream face of constant slope. They obtained the distribution of the hydrodynamic pressure along the sloping dam and presented explicit analytical formulas for evaluating the total horizontal, vertical and normal loads. They also pointed out an interesting feature that the total normal force acting on a dam was practically independent of the slope of the dam. In a subsequent paper, Chwang [2] presented an integral solution for the earthquake force on a rigid, sloping dam based on the exact, two-dimensional potential-flow theory. The results based on this exact theory were compared with those derived from the momentum-balance method. The two methods were found to be in reasonable agreement, especially for the total force exerted on the face of the dam.

The effect of finite reservoir on the hydrodynamic pressure was investigated by Werner and Sundquist [5]. They considered the motion of a fluid in basins of various cross sections, such as rectangular, semi-circular, and triangular (with a 90° vertex angle) cross sections. Their solutions contain expressions both for the dynamic water pressure and for the displacements in the fluid. For in-phase movement of the dam and the reservoir boundary, their solution indicates a decrease in the hydrodynamic pressure force on the dam surface due to the finite size of the reservoir.

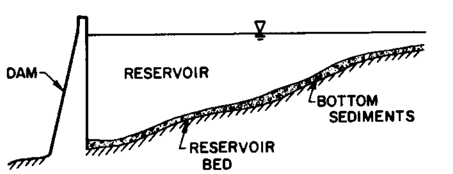

Fig. 1 A typical longitudinal section of a reservoir.

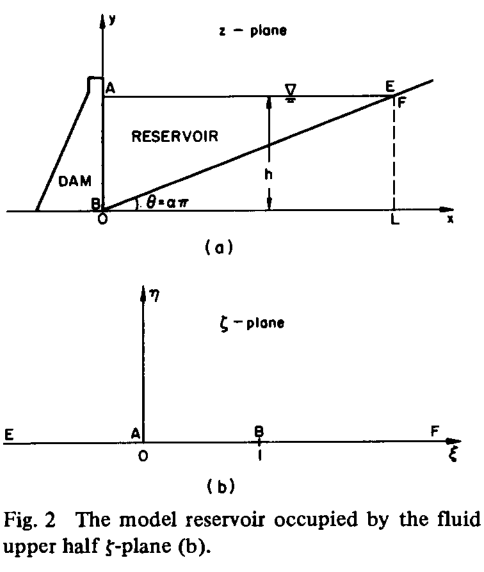

Fig. 2 The model reservoir occupied by the fluid in the physical z-plane (a) is mapped conformally into the upper half ζ-plane (b).

The objective of this paper is to present a solution for the hydrodynamic pressure force on a dam produced by the motion of water in a reservoir of more realistic shape (see Fig. 1) due to horizontal as well as vertical ground movements. The solution is derived from the exact, two-dimensional potential-flow theory. Possible cavitation at the upstream face of a dam when the dam accelerates away from the water is also discussed, and a simple criterion for determining the incipient cavitation is given.

2. Horizontal excitation

Figure 1 shows a typical longitudinal section of a reservoir. Neglecting the effect of bottom sediments on the motion of the fluid and approximating the reservoir bed by a straight line with an arbitrary angle $\theta(\theta = \alpha\pi)$ relative to the horizontal, one has an idealized model of a dam-reservoir system (see Fig. 2a) suitable for the following theoretical analysis. Let the x-axis be in the horizontal plane and perpendicular to the up-stream face of the dam which is assumed to be vertical. The y-axis is assumed to point vertically upwards and the still-water level in the reservoir is at $y = h$. The rigid dam and the reservoir bottom are assumed to have a constant horizontal acceleration a_h in the x-direction of sufficiently short duration so that the perturbation of the free surface is negligible.

With $z = x + iy$, the conformal mapping

$$z = \frac{h}{\sqrt{\pi}} \frac{\Gamma(1-\alpha)}{\Gamma(\frac{1}{2}-\alpha)} e^{i\alpha\pi} \int_1^{\zeta} \frac{d\zeta}{(\zeta-1)^{\alpha}\sqrt{\zeta(\zeta-1)}}, \tag{1}$$

given by the Schwarz-Christoffel theory, transforms the upper half ζ-plane ($\zeta = \xi + i\eta$) into the region occupied by the fluid in the reservoir (see Fig. 2). In equation (1), Γ represents the gamma function which is defined by the Eulerian integral of the second kind,

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt. \tag{2}$$

The points A and B in the physical z-plane are mapped into $\zeta = 0$ and $+1$ respectively. The points E and F at $z = L + ih$ are mapped into the points of infinity in the ζ-plane along the negative and positive real axis respectively, where L denotes the length of the reservoir measured along the free surface $y = h$,

$$L = h \cot \alpha\pi. \tag{3}$$

There are three branch points, namely the origin, point one, and the point of infinity, in the complex ζ-plane for the integrand in equation (1). The branch cut which connects these three branch points lies on the positive real axis from zero to infinity. The positive branch is taken for the square-root function.

Since the fluid in the reservoir is assumed to be incompressible and inviscid, the hydrodynamic pressure p (in excess of the hydrostatic pressure) produced by the horizontal acceleration a_h of the dam and of the reservoir bed satisfies the Laplace equation

$$\nabla^2 p = 0. \tag{4}$$

An analytic function f, regular in the upper half ζ-plane, can be formed by adding to p its complex conjugate function q as

$$f = p + iq, \tag{5}$$

where both p and q are real and q also satisfies the Laplace equation. On the free surface, which corresponds to the negative real axis in the ζ-plane, the pressure p vanishes. On the up-stream face of the dam the pressure gradient is a constant, $\partial p/\partial n = -\rho a_h$, where $\partial/\partial n$ denotes the normal derivative. Along the reservoir bed, the boundary condition requires that $\partial p/\partial n = \rho a_h \sin \alpha\pi$. Now, if s represents the distance measured along the boundary of the reservoir, which consists of the up-stream face of the dam AB and the reservoir bed BF, and $s = 0$ at $z = 0$, it follows from equation (1) that

$$s(\xi) = \frac{h}{\sqrt{\pi}} \frac{\Gamma(1-\alpha)}{\Gamma(\frac{1}{2}-\alpha)} \int_{\xi}^{1} \frac{d\xi}{(1-\xi)^{\alpha}\sqrt{\xi(1-\xi)}} \quad (0 < \xi < 1), \tag{6a}$$

$$= \frac{h}{\sqrt{\pi}} \frac{\Gamma(1-\alpha)}{\Gamma(\frac{1}{2}-\alpha)} \int_{\xi}^{1} \frac{d\xi}{(\xi-1)^{\alpha}\sqrt{\xi(\xi-1)}} \quad (1 < \xi < \infty). \tag{6b}$$

The Cauchy-Riemann condition for the analytic function $f = p + iq$ that $\partial p/\partial n = \partial q/\partial s$ gives $q = -\rho a_h s$ along AB and $q = \rho a_h s \sin \alpha\pi$ along BF. Therefore, the boundary conditions for $f(\zeta)$ along the real axis in the ζ-plane are

$$Re\, f(\zeta) = 0 \qquad (-\infty < \xi < 0), \tag{7a}$$

$$Im\, f(\zeta) = -\rho a_h s \qquad (0 < \xi < 1), \tag{7b}$$

$$Im\, f(\zeta) = \rho a_h s \sin \alpha\pi \qquad (1 < \xi < \infty), \tag{7c}$$

where s is given by equation (6), Re and Im denote the real and imaginary parts respectively.

In order to solve for $f(\zeta)$, it is convenient to introduce another analytic function $g(\zeta)$ by

$$g(\zeta) = \zeta^{-\frac{1}{2}} f(\zeta), \tag{8}$$

where the positive branch is taken for the square-root function. For this new function $g(\zeta)$ the boundary conditions (7a)–(7c) become

$$Im\, g(\zeta) = \begin{cases} 0 & (\zeta < 0), \\ -\rho a_h s(\zeta)\zeta^{-\frac{1}{2}} & (0 < \zeta < 1), \\ \rho a_h s(\zeta) \sin \alpha\pi \zeta^{-\frac{1}{2}} & (\zeta > 1), \end{cases} \tag{9}$$

for real ζ. An analytic function $g(\zeta)$, which is regular in the upper half ζ-plane and vanishes at infinity, can be obtained by the Poisson integral formula to give

$$g(\zeta) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{Im\, g(\xi) d\xi}{\xi - \zeta} . \tag{10}$$

Substituting equations (8) and (9) into (10), one has

$$f(\zeta) = -\frac{\rho h a_h \zeta^{\frac{1}{2}}}{\pi} \left[\int_0^1 \frac{s(\xi) d\xi}{\xi^{\frac{1}{2}} (\xi - \zeta)} - \sin \alpha\pi \int_1^{\infty} \frac{s(\xi) d\xi}{\xi^{\frac{1}{2}} (\xi - \zeta)} \right] . \tag{11}$$

The real part of $f(\zeta)$ for real ζ with $0 < \zeta < 1$ gives the hydrodynamic pressure on the dam, which may be obtained by substituting equations (6a) and (6b) into (11) to give

$$p(\xi) = \frac{\rho a_h \Gamma(1-\alpha)}{\pi^{3/2} \Gamma(\frac{1}{2} - \alpha)} \left[⨍_0^1 \log \left| \frac{t^{\frac{1}{2}} + \xi^{\frac{1}{2}}}{t^{\frac{1}{2}} - \xi^{\frac{1}{2}}} \right| \frac{dt}{(1-t)^{\alpha} \sqrt{t(1-t)}} \right.$$

$$\left. - \sin \alpha\pi \int_1^{\infty} \log \left| \frac{t^{\frac{1}{2}} + \xi^{\frac{1}{2}}}{t^{\frac{1}{2}} - \xi^{\frac{1}{2}}} \right| \frac{dt}{(t-1)^{\alpha} \sqrt{t(t-1)}} \right] \quad (0 < \xi < 1), \tag{12}$$

where $⨍$ denotes the Cauchy principal value. The second term in the square bracket of equation (12) may be simplified by differentiating it with respect to ξ as

$$- \sin \alpha\pi\, \xi^{-\frac{1}{2}} \int_1^{\infty} \frac{dt}{(t-1)^{\alpha + 1/2} (t - \xi)} ,$$

which can be reduced, through a contour integration in the complex t-plane around a branch cut between one and infinity and along a circle of large radius, to

$$- \frac{\pi \tan \alpha\pi}{(1-\xi)^{\alpha} \sqrt{\xi(1-\xi)}} .$$

Integrating the above with respect to ξ and making use of equation (6a), one obtains the pressure coefficient C_p,

$$C_p(\xi) = \frac{p(\xi)}{\rho a_h h} = \frac{\Gamma(1-\alpha)}{\pi^{3/2} \Gamma(\frac{1}{2} - \alpha)} ⨍_0^1 \log \left| \frac{t^{\frac{1}{2}} + \xi^{\frac{1}{2}}}{t^{\frac{1}{2}} - \xi^{\frac{1}{2}}} \right| \frac{dt}{(1-t)^{\alpha} \sqrt{t(1-t)}}$$

$$- \tan \alpha\pi \left[1 - \frac{s(\xi)}{h} \right] \qquad (0 < \xi < 1), \tag{13}$$

where $s(\xi)$ is given by equation (6a).

Along the up-stream face of the dam, the vertical height y is the same as the distance s measured from the bottom,

$$y(\xi) = s(\xi) \qquad (0 < \xi < 1). \tag{14}$$

Therefore the pressure distribution on the dam surface can be computed as a function of y with ξ being a parameter through equations (6a) and (13). Figure 3 shows the relationship between the pressure coefficient C_p and the normalized vertical distance y/h, as computed from equations (6a) and (13), for several fixed inclination angles of the reservoir bed θ ($\theta = \alpha\pi$) from 0° to 80°. It can be seen from Fig. 3 that for fixed height y/h, the pressure coefficient C_p decreases as θ increases; for fixed angle θ, C_p increases as y/h decreases and C_p reaches its maximum value at $y = 0$. In particular, for $\theta = 0°(\alpha = 0)$, equation (13) gives

$$C_p(1) = 8G/\pi^2 = 0.7425\,, \tag{15}$$

where $G = 0.915965$ ... is Catalan's constant. This result agrees exactly with the Westergaard [6] result for an infinitely long reservoir (see also Chwang [2]).

The total normal force on the dam can be found by integrating equation (13) in the physical z-plane as

$$F_h = \int_0^h p ds = \int_1^0 p(\xi) \frac{ds}{d\xi} d\xi, \tag{16}$$

or in the dimensionless form as

$$C_{hx} = \frac{F_h}{\rho a_h h^2} = \frac{\Gamma(1-\alpha)}{\sqrt{\pi}\,\Gamma(\frac{1}{2}-\alpha)} \int_0^1 \frac{C_p(\xi)d\xi}{(1-\xi)^\alpha\sqrt{\xi(1-\xi)}}\,, \tag{17}$$

where $C_p(\xi)$ is given by equation (13). The numerical result computed from equation (17) is shown in Fig. 4 in which the force coefficient due to horizontal acceleration, C_{hx}, is plotted versus the inclination angle of the reservoir bed, θ, or the reservoir depth-to-length ratio h/L. It

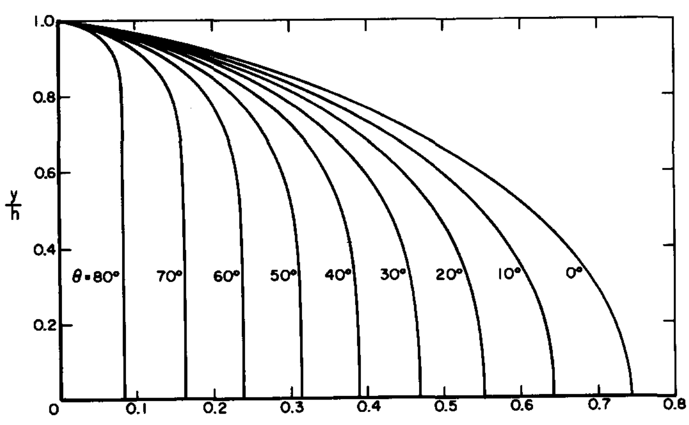

Fig. 3 The pressure distribution on the up-stream face of a dam for fixed inclination angle θ of the reservoir bed from 0° to 80°.

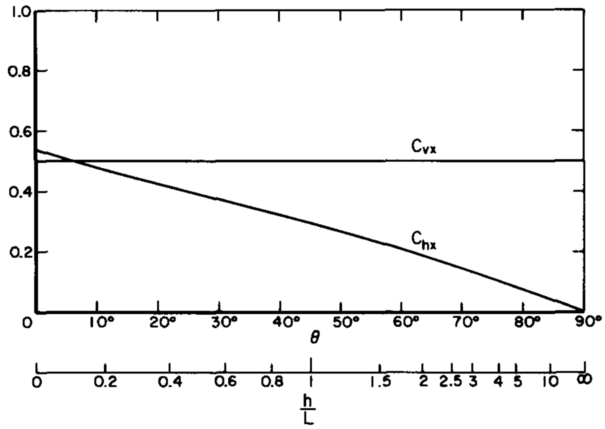

Fig. 4 The total hydrodynamic pressure forces C_{hx} and C_{vx}, due to horizontal and vertical accelerations respectively, versus the inclination angle θ of the reservoir bed or the reservoir depth-to-length ratio h/L.

is noted from Fig. 4 that for an infinitely long reservoir ($\theta = 0°$ or $h/L = 0$), C_{hx} reaches a maximum value of 0.543 which is the same as Westergaard's result. It starts to decrease almost linearly as θ increases, and it becomes zero when θ is 90° as it should be since there is no fluid in the reservoir then.

3. Vertical excitation

If the rigid dam and the reservoir bed experience a constant vertical acceleration a_v in the y-direction during an earthquake, then the boundary conditions for the hydrodynamic pressure p become (see Fig. 2a)

$$\frac{\partial p}{\partial x} = 0 \qquad (x = 0), \tag{18a}$$

$$\sin \alpha\pi \, \frac{\partial p}{\partial x} - \cos \alpha\pi \, \frac{\partial p}{\partial y} = \rho a_v \cos \alpha\pi \quad (y = x \tan \alpha\pi), \tag{18b}$$

$$p = 0 \qquad (y = h). \tag{18c}$$

The solution of equation (4) satisfying the above boundary conditions (18a)–(18c) is simply

$$p = \rho a_v (h - y), \tag{19}$$

which has the same expression as the hydrostatic pressure except that the gravitational constant g is replaced by the constant vertical acceleration a_v. Therefore the effect of vertical acceleration on the pressure force acting on a dam is simply to adjust the hydrostatic pressure by replacing the gravitational constant g by an effective gravitational acceleration $(g + a_v)$.

The total normal force on the dam is obtained by integrating equation (19),

$$F_v = \int_0^h p dy = \tfrac{1}{2}\, \rho a_v h^2 . \tag{20}$$

Thus the dimensionless force coefficient C_{vx} becomes

$$C_{vx} = \frac{F_v}{\rho a_v h^2} = 0.5 , \tag{21}$$

which is independent of the inclination angle of the reservoir bed θ. The value of C_{vx} is also shown in Fig. 4 for comparison. It can be seen from Fig. 4 that if the vertical acceleration a_v is of the same magnitude as the horizontal acceleration a_h, the total hydrodynamic pressure force due to a_v would be greater than that due to a_h for $\theta > 7°$. Hence the hydrodynamic effect due to the vertical excitation during an earthquake may be equally important as that due to the horizontal component of the ground acceleration.

It should be noted that if the up-stream face of the dam and the reservoir bed take any arbitrary configuration other than that shown in Fig. 2a, the hydrodynamic pressure due to vertical ground acceleration a_v would still be given by equation (19) since it satisfies the boundary conditions

$$p = 0 \qquad (y = h), \tag{22a}$$

$$(\vec{n} \cdot \nabla) p = - \rho a_v n_y \qquad (\text{on } S), \tag{22b}$$

where S denotes the boundary of the reservoir and $\vec{n} = (n_x, n_y)$ is the outward normal vector along S.

4. Possible cavitation

If the rigid dam together with the reservoir bed moves with a constant horizontal acceleration a_h in the negative x-direction (away from the water) or with a constant vertical acceleration a_v in the negative y-direction (see Fig. 2a), then the resulting hydrodynamic pressure p would still be given by equation (13) or (19) respectively but with a sign change, that is, the hydrodynamic pressure becomes negative, since the problem is linear. Should the hydrodynamic pressure become negatively so large that the total absolute pressure (hydrodynamic plus hydrostatic and atmospheric pressure) becomes negative, cavitation could take place at the up-stream face of the dam because the water in the reservoir cannot sustain any tension. To see that cavitation is possible, it is convenient, without loss of generality, to consider the case in which the reservoir is infinitely long ($\alpha\pi = 0$) and the dam has a constant acceleration a_h in the negative x-direction. For the present purpose it is quite sufficient to approximate the hydrodynamic pressure by the result derived from the momentum balance method (see von Kármán [4] and Chwang [2]). Thus

$$p = - \frac{1}{\sqrt{2}} \rho a_h h \sqrt{1 - \left(\frac{y}{h}\right)^2} . \tag{23}$$

Denoting the horizontal acceleration a_h by

$$a_h = kg\,, \tag{24}$$

where k is the earthquake intensity, one has the total absolute pressure p_{total},

$$p_{\text{total}} = -\frac{1}{\sqrt{2}}\,\rho kgh\sqrt{1-\left(\frac{y}{h}\right)^2} + \rho gh\left[1-\frac{y}{h}\right] + p_a\,, \tag{25}$$

where p_a is the atmospheric pressure.

The criterion for cavitation to occur is that

$$p_{\text{total}} < 0\,. \tag{26}$$

It follows from equations (25) and (26) that

$$\frac{k}{\sqrt{2}}\sqrt{1-\left(\frac{y}{h}\right)^2} + \frac{y}{h} > 1 + \frac{p_a}{\rho gh}\,. \tag{27}$$

The left-hand side of the above inequality has a maximum value of $\sqrt{1+(k^2/2)}$ at $y/h = \sqrt{2/(2+k^2)}$ since its first derivative with respect to y vanishes and its second derivative with respect to y is negative there. Therefore, in order to assure that cavitation will take place on the up-stream face of a dam one must require

$$\sqrt{1+\frac{k^2}{2}} > 1 + \frac{p_a}{\rho gh}\,. \tag{28}$$

Morris Dam, located about 10 miles from Pasadena, California, is a large concrete gravity dam with a height of 328 ft. This dam could serve as an example for numerical calculations. For estimates, the following values are taken: $p_a = 14.7 \times 144 \times 32.2$ poundals/ft^2 (1.01×10^6 dynes/cm^2); $\rho = 62.4$ pounds/ft^3 (1.00 gram/cm^3); $g = 32.2$ ft/sec^2 (981 cm/sec^2); and $h = 328$ ft (1.00×10^4 cm) so that $p_a/(\rho gh) = 0.103$. It then follows from the inequality (28) that

$$k > 0.658\,.$$

The above value for the earthquake intensity falls within the range of real earthquakes in Southern California. Detailed calculation on incipient cavitation and its damage on the up-stream face of a dam will be presented in future papers.

5. Conclusions

The effect of finite reservoir on the hydrodynamic pressure due to horizontal as well as vertical ground excitations has been studied. It is found that for horizontal accelerations the hydrodynamic pressure force decreases as the size of the reservoir decreases. The effect of vertical acceleration on the pressure force on a dam is simply to adjust the hydrostatic pressure by replacing the gravitational constant by an effective gravitational acceleration and this is true for any arbitrary shapes of the reservoir. If the vertical acceleration is of the same order of magnitude as the horizontal acceleration, it will cause the water to exert a hydrodynamic pressure force on a dam with a vertical up-stream face as large as that due to the horizontal acceleration.

A simple criterion has been presented in this paper which would enable dam engineers to determine whether a given earthquake will cause cavitation at the dam-water interface or not.

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